

## LOCAL INVARIANTS OF ISOGENOUS ELLIPTIC CURVES

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ABSTRACT. We investigate how various invariants of elliptic curves, such as the discriminant, Kodaira type, Tamagawa number and real and complex periods, change under an isogeny of prime degree  $p$ . For elliptic curves over  $l$ -adic fields, the classification is almost complete (the exception is wild potentially supersingular reduction when  $l = p$ ), and is summarised in a table.

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## 1. INTRODUCTION

We address the question how various invariants of elliptic curves, such as the discriminant, Kodaira type, Tamagawa number and real and complex periods, change under an isogeny. As every isogeny factors as a composition of endomorphisms and isogenies of prime degree, throughout the paper we just consider a fixed isogeny

$$\phi : E \longrightarrow E'$$

of prime degree  $p$ . Our first result relates the discriminants  $\Delta_E$  and  $\Delta_{E'}$ :

**Theorem 1.1.** *Let  $\mathcal{K}$  be a field of characteristic 0 and  $\phi : E \rightarrow E'$  a  $p$ -isogeny of elliptic curves over  $\mathcal{K}$ . If  $p > 3$ , then  $\Delta_E^p / \Delta_{E'}$  is a 12th power in  $\mathcal{K}$ . For  $p = 2, 3$  this is a 3rd, respectively 4th power.*

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The theorem is a simple consequence of the fact that  $\Delta(p\tau)^p/\Delta(\tau)$  is a 12th power of a modular form on  $\Gamma_0(p)$  for  $p > 3$ ; see Theorems 2.1, 2.2, 2.3 and §10. An alternative proof can be extracted from the appendix in [3]. For one application, see Česnavičius' work on the parity conjecture [2] §5.

Then we consider the standard invariants over local fields. The following table summarises our results for the valuations of minimal discriminants  $\delta, \delta'$  of  $E, E'$ , their Tamagawa numbers  $c, c'$ , Kodaira types and the leading term  $\frac{\phi^*\omega'}{\omega}$  of  $\phi$  on the formal group:

Reduction type of $E/K$	$\delta, \delta'$	$\frac{\phi^*\omega'}{\omega}$ up to unit	$\frac{c}{c'}$	Kodaira types for $E, E'$
good ordinary good supersingular	$\delta = \delta' = 0$	1 or $p$ ( $\dagger$ ) ?	1	$I_0$
split mult., $v(j) = pv(j')$ split mult., $pv(j) = v(j')$	$\delta = p\delta'$ $\delta' = p\delta$	1 $p$	$p$ $\frac{1}{p}$	$I_{pn}, I_n$ $I_n, I_{pn}$
nonsplit mult., $v(j) = pv(j')$	$\delta = p\delta'$	1	1 if $p \neq 2$ or $2 \delta'$ 2 otherwise	$I_{pn}, I_n$
nonsplit mult., $pv(j) = v(j')$	$\delta' = p\delta$	$p$	1 if $p \neq 2$ or $2 \delta$ $\frac{1}{2}$ otherwise	$I_n, I_{pn}$
additive pot. mult. $v(j) = pv(j')$	$\delta' = \delta + \frac{p-1}{p}v(j)$	1	1 if $p \geq 3$ ( $\dagger$ ) if $p = 2$	$I_n^*, I_{n+\frac{p-1}{p}v(j)}^*$ ( $= I_{n/p}^*$ if $l \neq 2$ )
$pv(j) = v(j')$	$\delta' = \delta - (p-1)v(j)$	$p$	1 if $p \geq 3$ ( $\dagger$ ) if $p = 2$	$I_n^*, I_{n-(p-1)v(j)}^*$ ( $= I_{pn}^*$ if $l \neq 2$ )
additive pot. good, $l \neq p$ $p = 3$ , type IV, IV*, $\mu_3 \not\subset K$ $p = 2$ , type $I_0^*$ all other cases	$\delta' = \delta$	1	(*) ( $\dagger$ ) 1	same
additive pot. good, $l = p$ pot. ordinary	$\delta' = \delta$	1 or $p$ ( $\dagger$ )	1 if $p \geq 3$ ( $\dagger$ ) if $p = 2$	same
pot. supersingular tame	$\delta' = 12 - \delta$	?	1	opposite
pot. supersingular wild	?	?	?	?
$K/\mathbb{Q}_l$ finite, $v: K^\times \rightarrow \mathbb{Z}$ valuation; $\phi: E/K \rightarrow E'/K$ $p$ -isogeny; $\Delta, \Delta'$ minimal discriminants; $\delta = v(\Delta), \delta' = v(\Delta')$ ; $\omega, \omega'$ minimal differentials; $j, j'$ $j$ -invariants; $c, c'$ Tamagawa numbers.				
(*) = 3 if $E/K$ has non-trivial 3-torsion, and $\frac{1}{3}$ otherwise.				
$(\dagger) = \begin{cases} 1 & \text{if } \frac{\Delta}{\Delta'} \text{ is a norm in } F/K \\ \frac{1}{2} & \text{if } \Delta' \text{ is a norm in } F/K \text{ and } \Delta \text{ is not} \\ 2 & \text{if } \Delta \text{ is a norm in } F/K \text{ and } \Delta' \text{ is not} \end{cases}$ $(\dagger) = \begin{cases} p & \text{if } \ker \phi \subset \hat{E}(\mathfrak{m}_L) \\ 1 & \text{if } \ker \phi \not\subset \hat{E}(\mathfrak{m}_L) \end{cases} \quad (L = F(\ker \phi))$				
$F$ is any ( $\dagger$ ), respectively quadratic ( $\dagger$ ), extension where $E$ has good or split multiplicative reduction.				

Table 1. Local invariants of isogenous elliptic curves

The quotient of Tamagawa numbers  $\frac{c}{c'}$  and the quantity  $\frac{\phi^*\omega'}{\omega}$  classically appear in the applications of the isogeny invariance of the Birch–Swinnerton-Dyer formula to Selmer groups of elliptic curves, see e.g. [1], [15], [8], [5], [10] and [7]. The quotient  $\frac{\phi^*\omega'}{\omega}$  is an important invariant of the isogeny  $\phi$ , being the leading term of  $\phi$  on the formal groups (cf. Lemma 4.2, [19] §IV.4 and also [15] p.91, where it is denoted by  $\phi'(0)$ ).

For curves defined over  $\mathbb{R}$  and  $\mathbb{C}$  the analogues of the local Tamagawa numbers are periods (cf. Remark 7.5)

$$\Omega(E, \omega) = \int_{E(\mathbb{R})} |\omega| \quad \text{and} \quad \Omega(E, \omega) = 2 \int_{E(\mathbb{C})} \omega \wedge \bar{\omega}$$

computed with respect to some invariant differential  $\omega$  on  $E$ . We show that these periods for  $E$  and  $E'$  are related as follows:

**Theorem 1.2.** *Suppose the base field of  $E, E'$  is  $\mathcal{K} = \mathbb{R}$  or  $\mathcal{K} = \mathbb{C}$ . Choose invariant differentials  $\omega, \omega'$  for  $E$  and  $E'$ . Then*

$$\frac{\Omega(E, \omega)}{\Omega(E', \omega')} = \lambda \left| \frac{\omega}{\phi^*\omega'} \right|_{\mathcal{K}}.$$

Here  $|\cdot|_{\mathcal{K}}$  is the standard normalised absolute value on  $\mathcal{K}$ , and  $\lambda$  is

- $p$  if  $\mathcal{K} = \mathbb{C}$ ,
- $p$  if  $\mathcal{K} = \mathbb{R}$ ,  $p \neq 2$  and  $\ker \phi \subset E(\mathbb{R})$ ,
- $1$  if  $\mathcal{K} = \mathbb{R}$ ,  $p \neq 2$  and  $\ker \phi \not\subset E(\mathbb{R})$ .

If  $\mathcal{K} = \mathbb{R}$  and  $p = 2$ , write  $E$  in the form  $y^2 = x^3 + ax^2 + bx$  so that  $(0, 0) \in \ker \phi$ . Then  $\lambda$  is

- $1$  if  $b > 0$ , and either  $a < 0$  or  $4b > a^2$ ,
- $2$  otherwise.

Finally, we look at periods of isogenous elliptic curves over  $\mathbb{Q}$ . In this case,  $E/\mathbb{Q}$  and  $E'/\mathbb{Q}$  have global minimal differentials  $\omega, \omega'$ , unique up to signs. The real periods  $\Omega = \Omega(E/\mathbb{R}, \omega)$  and  $\Omega' = \Omega(E'/\mathbb{R}, \omega')$  are the ones that enter the Birch–Swinnerton-Dyer conjecture over  $\mathbb{Q}$ . We prove that the quotient  $\Omega/\Omega'$  is  $1, p$  or  $1/p$  and give a criterion for when it is  $1$  (Theorem 8.2). E.g., for  $p > 3$  the periods are equal if and only if  $E$  has an odd number of primes of additive reduction with local root number  $-1$ . If  $E$  is semistable, we establish a stronger statement (Theorem 8.7):

**Theorem 1.3.** *If  $\phi : E \rightarrow E'$  is a  $p$ -isogeny of semistable elliptic curves over  $\mathbb{Q}$  with  $p$  odd, then  $\frac{\Omega}{\Omega'} = p^{\pm 1}$ . Moreover,*

$$\frac{\Omega}{\Omega'} = p \iff \ker \phi \subset E(\mathbb{Q}),$$

which is also equivalent to  $\omega = \pm \phi^*\omega'$ .

**1.1. Notation.** Throughout the paper  $p$  is a prime number, and  $\phi : E \rightarrow E'$  an isogeny of elliptic curves of degree  $p$ . We write  $\phi^t : E' \rightarrow E$  for the dual isogeny. In §3–§6, the base field  $K$  is a finite extension of  $\mathbb{Q}_l$ ;  $l = p$  is allowed.

There we use the following notation:

$v$	normalised valuation $K^\times \rightarrow \mathbb{Z}$
$\mathfrak{m}_K$	maximal ideal of the ring of integers of $K$
$\Delta, \Delta'$	minimal discriminants of $E/K$ and $E'/K$
$\delta, \delta'$	their valuations: $\delta = v(\Delta), \delta' = v(\Delta')$
$\omega, \omega'$	minimal invariant differentials on $E, E'$ (unique up to units)
$f = f'$	conductor exponent of $E$ and $E'$
$c, c'$	local Tamagawa numbers of $E$ and $E'$
$m, m'$	number of components in the special fibre of the Néron models of $E, E'$ ; so $\delta = f + m - 1, \delta' = f' + m' - 1$ by Ogg's formula
$j, j'$	$j$ -invariants of $E$ and $E'$
$\hat{E}, \hat{E}'$	the formal groups of $E$ and $E'$ .

When it is necessary to work over an extension  $F/K$ , we write  $\Delta_{E/F}, \Delta_{E'/F}$  etc.

Recall that a curve  $E/K$  has additive reduction if and only if it has conductor exponent  $f \geq 2$ , and  $f = 2$  if and only if the  $\ell$ -adic Tate module of  $E$  is tamely ramified for some (any)  $\ell \neq l$ . We will call this *tame* reduction (and *wild* otherwise). If  $l \geq 5$ , the reduction is always tame; when  $l = 2$  it is tame if and only if  $E$  has Kodaira type IV, IV\*; when  $l = 3$  it is tame if and only if  $E$  has Kodaira type III, III\* or  $I_0^*$  (cf. Theorem 3.1). In Table 1, *opposite* Kodaira types refers to  $\text{II} \leftrightarrow \text{II}^*, \text{III} \leftrightarrow \text{III}^*, \text{IV} \leftrightarrow \text{IV}^*, I_0^* \leftrightarrow I_0^*$ .

**1.2. Layout.** Theorem 1.1 is proved in §2. In §3-§6 we prove the results summarised in Table 1: for  $\delta$  see 5.1 and 3.3; for  $\phi^*\omega'/\omega$  see 4.8, 4.10, 4.3 and 4.9; for the Tamagawa numbers, see 6.1; for Kodaira symbols, see 5.4. Real and complex periods are discussed in §7 and the particular case of elliptic curves over  $\mathbb{Q}$  in §8. The appendix in §9 recalls the theory of the Tate curve and some standard facts about quadratic twists. The appendix in §10 reviews the  $q$ -expansion principle and the connection between values of modular forms and invariants of elliptic curves.

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## 2. $\Delta(E') = \Delta(E)^p$ UP TO 12TH POWERS

In this section we relate the discriminants  $\Delta$  and  $\Delta'$  of  $p$ -isogenous elliptic curves  $E$  and  $E'$ . Recall that the discriminant depends on the choice of a Weierstrass model, and is well-defined up to 12th powers.

**Theorem 2.1.** *Let  $\mathcal{K}$  be a field of characteristic 0 and  $\phi : E \rightarrow E'$  a  $p$ -isogeny of elliptic curves over  $\mathcal{K}$  with  $p > 3$ . Then  $\Delta^p/\Delta'$  is a 12th power in  $\mathcal{K}$ .*

*Proof.* (See also [3], appendix, for an alternative approach.) We will assume that  $\mathcal{K} \subset \mathbb{C}$ , embedding the field of definition of  $E, E'$  and  $\phi$  into  $\mathbb{C}$  if necessary.

Let  $\tau$  be a complex variable in the upper-half plane, and let  $\eta(\tau)$  be the Dedekind eta-function. By a theorem of Klyachko [11] Thm 2.2,  $\eta(p\tau)^p/\eta(\tau)$  is a modular form of weight  $\frac{p-1}{2}$  on  $\Gamma_0(p)$ , with character  $(\frac{d}{p})$ . Its square  $f(\tau) = [\eta(p\tau)^p/\eta(\tau)]^2$  is a modular form of weight  $p-1$  on  $\Gamma_0(p)$ , with trivial character. Its  $q$ -expansion

$$f(\tau) = q^{\frac{p^2-1}{12}} \prod_{n \geq 1} \frac{(1 - q^{pn})^{2p}}{(1 - q^n)^2} \quad (q = e^{2\pi i \tau}),$$

clearly has integer coefficients. Note that  $f(\tau)$  is a 12th root of  $\frac{\Delta(p\tau)^p}{\Delta(\tau)}$ .

Choose complex uniformisations  $E = \mathbb{C}/\Lambda$ ,  $E' = \mathbb{C}/\Lambda'$  with  $\Lambda = \mathbb{Z}\Omega_1 + \mathbb{Z}\Omega_2$ ,  $\Lambda' = \mathbb{Z}\Omega_1 + \mathbb{Z}p\Omega_2$  as in §10. By the  $q$ -expansion principle (Theorem 10.3),

$$\left(\frac{2\pi}{\Omega_1}\right)^{p-1} f(\tau) \in \mathcal{K}, \quad \tau = \frac{\Omega_2}{\Omega_1}.$$

On the other hand, by (10.2),

$$f(\tau)^{12} = \frac{\Delta(p\tau)^p}{\Delta(\tau)} = \frac{\left(\frac{\Omega_1}{2\pi}\right)^{12p} \Delta_{E'}^p}{\left(\frac{\Omega_1}{2\pi}\right)^{12} \Delta_E},$$

where  $\Delta_E$  and  $\Delta_{E'}$  are discriminant of two specific  $\mathcal{K}$ -rational models, cf. (10.2). It follows that

$$\frac{\Delta_{E'}^p}{\Delta_E} = \left[\left(\frac{2\pi}{\Omega_1}\right)^{p-1} f(\tau)\right]^{12} \in \mathcal{K}^{\times 12}.$$

Swapping  $E$  and  $E'$  (or using the fact that  $(\Delta^p)^p$  is  $\Delta$  up to a 12th power, as  $p^2 \equiv 1 \pmod{12}$  for  $p \neq 2, 3$ ) gives the claim.  $\square$

Now we prove analogues for  $p = 2$  and  $p = 3$ , using an explicit computation with a universal family:

**Theorem 2.2.** *Let  $\mathcal{K}$  be a field of characteristic 0 and  $\phi : E \rightarrow E'$  a 2-isogeny of elliptic curves over  $\mathcal{K}$ . Then  $\Delta^2/\Delta'$  is a 3rd power in  $\mathcal{K}$ .*

*Proof.* Any 2-isogeny  $\phi : E \rightarrow E'$  of elliptic curves over a field of characteristic not 2 or 3 has a model

$$\begin{aligned} E &: y^2 = x^3 + ax^2 + bx, \\ E' &: y^2 = x^3 - 2ax^2 + (a^2 - 4b)x, \\ \phi(x, y) &= (x + a + bx^{-1}, y - byx^{-2}), \end{aligned}$$

and

$$\frac{\Delta^2}{\Delta'} = \frac{[16b^2(a^2 - 4b)]^2}{256b(a^2 - 4b)^2} = b^3.$$

$\square$

**Theorem 2.3.** *Let  $\mathcal{K}$  be a field of characteristic 0 and  $\phi : E \rightarrow E'$  a 3-isogeny of elliptic curves over  $\mathcal{K}$ . Then  $\Delta^3/\Delta'$  is a 4th power in  $\mathcal{K}$ .*

*Proof.* Any 3-isogeny  $\phi : E \rightarrow E'$  of elliptic curves over a field of character-

istic not 2 or 3 has a model

$$\begin{aligned} E &: y^2 = x^3 + a(x-b)^2, \\ E' &: y^2 = x^3 + ax^2 + 18abx + ab(16a - 27b), \\ \phi(x, y) &= (x - 4abx^{-1} + 4ab^2x^{-3}, y + 4abyx^{-2} - 8ab^2yx^{-3}), \end{aligned}$$

and

$$\frac{\Delta^3}{\Delta'} = \frac{[-16a^2b^3(4a + 27b)]^3}{-16a^2b(4a + 27b)^3} = (4ab^2)^4.$$

□

### 3. DISCRIMINANTS AND KODAIRA TYPES I

Throughout §3-§6 we follow the notation of §1.1. In particular,  $K$  is a finite extension of  $\mathbb{Q}_l$ , and  $\phi : E/K \rightarrow E'/K$  is a  $p$ -isogeny.

**Theorem 3.1.** *Suppose  $E/K$  has additive potentially good reduction. Then  $E$  has tame reduction (equivalently, has conductor exponent 2) if and only if*

- $l \geq 5$ , or
- $l = 3$  and  $E$  has Kodaira type III, III\*, I<sub>0</sub>\*, or
- $l = 2$  and  $E$  has Kodaira type IV, IV\*.

*In this case  $E'$  is tame as well, and*

$$0 < \delta, \delta' < 12 \quad \text{and} \quad \delta' \equiv p\delta \pmod{12}.$$

*Proof.* For the first statement, see [20] IV.9, Table 4.1 for  $l \geq 5$ , [13] Thm. 1 for  $l = 3$ , and [12] Prop 8.20 for  $l = 2$ . From [20] IV.9, Table 4.1 it also follows that  $0 < \delta, \delta' < 12$  in all these cases. The last congruence follows from Theorems 2.1–2.3. □

**Theorem 3.2.** *Suppose  $E/K$  has additive potentially good reduction and is not a quadratic twist of a curve with good reduction. Then*

- *If  $l = 2, 3$  or  $l \equiv -1 \pmod{12}$  then  $E$  is potentially supersingular.*
- *If  $l \equiv 1 \pmod{12}$ , then  $E$  is potentially ordinary.*
- *If  $l \equiv 5 \pmod{12}$ , then  $E$  is potentially ordinary if and only if its Kodaira type is III or III\*.*
- *If  $l \equiv 7 \pmod{12}$ , then  $E$  is potentially ordinary if and only if its Kodaira type is II, II\*, IV or IV\*.*

*Proof.* Let  $K^{nr}$  be the maximal unramified extension of  $K$ , and  $F/K^{nr}$  the (finite) extension cut out by the Galois action on any  $\ell$ -adic Tate module of  $E$  for  $\ell \neq l$ . By the criterion of Néron-Ogg-Shafarevich,  $F$  is the unique minimal Galois extension of  $K^{nr}$  where  $E$  has good reduction.

The Galois group  $\text{Gal}(F/K^{nr})$  has order at least 3, since  $E$  is not a quadratic twist of a curve with good reduction (cf. Lemma 9.3). As explained in [18] proof of Thm. 2, it acts faithfully on the reduced curve  $\tilde{E}$  defined over the residue field of  $F$  as a group of automorphisms. This forces  $j(\tilde{E})$  to be either 0 or 1728, see e.g. [19] Thm. III.10.1. If  $l = 2$  or 3, then  $1728 = 0$  is a supersingular  $j$ -invariant, as asserted.

Now suppose  $l > 3$ . By [20] IV.9, Table 4.1, either  $E$  has reduction type II, II\*, IV or IV\* and  $j(E)$  reduces to 0, or reduction type III, III\* and  $j(E)$  reduces to 1728. The  $j$ -invariant 0 is ordinary if and only if  $l \equiv 1 \pmod{3}$ , and 1728 is ordinary if and only if  $l \equiv 1 \pmod{4}$ ; see [19] Ex. V.4.4, V.4.5.  $\square$

**Corollary 3.3.** *Suppose  $l = p$  and  $E$  has additive tame potentially good reduction. If the reduction is potentially ordinary, then  $\delta = \delta'$  and  $E, E'$  have the same Kodaira type. If the reduction is potentially supersingular, then  $\delta = 12 - \delta'$  and  $E, E'$  have opposite Kodaira types (II  $\leftrightarrow$  II\*, III  $\leftrightarrow$  III\*, IV  $\leftrightarrow$  IV\*,  $I_0^* \leftrightarrow I_0^*$ ).*

*Proof.* By Theorem 3.1, we have  $\delta, \delta' < 12$  and, if  $\delta \neq \delta'$ , then  $\delta \not\equiv p\delta \pmod{12}$ , equivalently  $12 \nmid \delta(p-1)$ . Exchanging  $E, E'$  if necessary, the possibilities with  $\delta \neq \delta'$  are (cf. [20] IV.9, Table 4.1)

- $\delta = 2, \delta' = 10, p \equiv 2 \pmod{3}$ ,
- $\delta = 4, \delta' = 8, p \equiv 2 \pmod{3}$ ,
- $\delta = 3, \delta' = 9, p \equiv 3 \pmod{4}$ .

By Theorem 3.2, these are precisely the cases of potentially supersingular reduction unless  $E$  is quadratic twist of a curve with good reduction. In the latter case,  $l$  cannot be 2 (as  $E$  has tame reduction), so  $E$  and  $E'$  have Kodaira type  $I_0^*$  and  $\delta = \delta' = 6$ .  $\square$

#### 4. DIFFERENTIALS

**Notation 4.1.** We will write

$$\alpha_{E/K} = \left| \frac{\phi^* \omega'}{\omega} \right|_K^{-1},$$

where  $|\cdot|_K$  is the normalised absolute value on  $K$ .

**Lemma 4.2.**

- (1) *The isogeny  $\phi$  induces a map on formal groups,*

$$\phi : \hat{E}(\mathfrak{m}_K) \rightarrow \hat{E}'(\mathfrak{m}_K), \quad \phi(T) = aT + \cdots,$$

*with leading term  $a = \frac{\phi^* \omega'}{\omega} \cdot \text{unit}$ .*

- (2)

$$\frac{|\text{coker } \phi : E(K) \rightarrow E'(K)|}{|\text{ker } \phi : E(K) \rightarrow E'(K)|} = \alpha_{E/K} \frac{c'}{c}.$$

*Proof.* (1) [19] Cor IV.4.3; (2) [15] Lemma 3.8.  $\square$

**Lemma 4.3.** *If  $l \neq p$ , then  $\phi^* \omega'$  is minimal, so  $\alpha_{E/K} = 1$ .*

*Proof.* Write  $\phi^* \omega' = a\omega$ ,  $(\phi^t)^* \omega = a' \omega'$ . Since the pullback of an integral differential is integral, we must have  $a, a' \in \mathcal{O}_K$ . On the other hand,  $\phi^t \phi = [p]$  implies  $aa' = p \in \mathcal{O}_K^\times$ . So  $a$  and  $a'$  are units, and  $\phi^* \omega'$  is minimal.  $\square$

**Lemma 4.4.** *Suppose  $F/K$  is a finite extension. Then*

$$\frac{\phi^* \omega_{E'/K}}{\omega_{E/K}} = \frac{\phi^* \omega_{E'/F}}{\omega_{E/F}} \times \text{unit} \quad \Longleftrightarrow \quad \frac{\Delta_{E'/K}}{\Delta_{E'/F}} = \frac{\Delta_{E'/F}}{\Delta_{E'/F}} \times \text{unit}.$$

*If  $l \neq p$ , or  $E/K$  is semistable, or  $l = p$  and  $E$  has tame potentially ordinary reduction, then the formulae hold.*

*Proof.* It is easy to see that up to units (cf. [19] Table III.1.2),

$$\frac{\Delta_{E/K}}{\Delta_{E/F}} = \left( \frac{\omega_{E/K}}{\omega_{E/F}} \right)^{-12} \quad \text{and} \quad \frac{\Delta_{E'/K}}{\Delta_{E'/F}} = \left( \frac{\omega_{E'/K}}{\omega_{E'/F}} \right)^{-12} = \left( \frac{\phi^* \omega_{E'/K}}{\phi^* \omega_{E'/F}} \right)^{-12}.$$

So  $\frac{\Delta_{E/K}}{\Delta_{E'/K}} / \frac{\Delta_{E/F}}{\Delta_{E'/F}}$  is the 12th power of  $\frac{\phi^* \omega_{E'/F}}{\omega_{E/F}} / \frac{\phi^* \omega_{E'/K}}{\omega_{E/K}}$ , up to a unit.

For the second claim, if  $l \neq p$  or  $E/K$  is semistable, then the left-hand formula holds (Lemma 4.3 and the fact that for semistable curves minimal differentials stay minimal in all extensions). If  $l = p$  and  $E$  is tame, the right-hand formula holds by Corollary 3.3.  $\square$

**Remark 4.5.** Suppose  $E$  and  $E'$  are in Weierstrass form,

$$E : y^2 = f(x), \quad E' : y^2 = g(x).$$

Since  $\phi(-P) = -\phi(P)$  and every even rational function on  $E$  is a function of  $x$  (cf. [19], proof of Cor. III.2.3.1),  $\phi$  has the form

$$\phi : (x, y) \mapsto (\xi(x), y\eta(x)), \quad \xi(x), \eta(x) \in K(x).$$

If  $F = K(\sqrt{d})$  is a quadratic extension, and

$$E_d : dy^2 = f(x), \quad E'_d : dy^2 = g(x)$$

the quadratic twists of  $E, E'$  by  $d$ , then the same formula  $(\xi(x), y\eta(x))$  defines an isogeny  $\phi_d : E_d \rightarrow E'_d$ . It fits into a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & E_d(K) & \longrightarrow & E(F) & \xrightarrow{N} & E(K) \longrightarrow \frac{E(K)}{NE(F)} \longrightarrow 0 \\ & & \phi_d \downarrow & & \phi \downarrow & & \phi \downarrow \\ 0 & \longrightarrow & E'_d(K) & \longrightarrow & E'(F) & \xrightarrow{N} & E'(K) \longrightarrow \frac{E'(K)}{NE'(F)} \longrightarrow 0, \end{array}$$

where the map  $E_d(K) \rightarrow E(F)$  is  $(x, y) \mapsto (x, y\sqrt{d})$ , and  $N$  is the norm (or trace) map  $E(F) \rightarrow E(K)$ ,  $E'(F) \rightarrow E'(K)$ .

**Lemma 4.6.** *Let  $F = K(\sqrt{d})$  be a quadratic extension,  $E_d, E'_d$  the quadratic twists of  $E, E'$  by  $d$ , and  $\phi_d$  the corresponding isogeny. The groups  $\frac{E(K)}{NE(F)}$ ,  $\frac{E'(K)}{NE'(F)}$  are finite, and*

$$\left( \alpha_{E_d/K} \frac{c_{E'_d/K}}{c_{E_d/K}} \right)^{-1} \cdot \alpha_{E/F} \frac{c_{E'/F}}{c_{E/F}} \cdot \left( \alpha_{E/K} \frac{c_{E'/K}}{c_{E/K}} \right)^{-1} \cdot \frac{|\frac{E'(K)}{NE'(F)}|}{|\frac{E(K)}{NE(F)}|} = 1.$$

*Proof.* The groups  $\frac{E(K)}{NE(F)}$ ,  $\frac{E'(K)}{NE'(F)}$  are quotients of  $\frac{E(K)}{2E(K)}$ ,  $\frac{E'(K)}{2E'(K)}$ , which are



finite. Now consider the commutative diagram above. Because the alternating product of  $|\ker|/|\text{coker}|$  is 1, Lemma 4.2(2) gives the claim.  $\square$

**Proposition 4.7.** *Let  $F = K(\sqrt{d})$  be a quadratic extension,  $E_d, E'_d$  the quadratic twists of  $E, E'$  by  $d$ , and  $\phi_d$  the corresponding isogeny.*

(1) *Write  $K_n, F_n$  for the degree  $n$  unramified extensions of  $K, F$ . Then*

$$\frac{\alpha_{E/K} \alpha_{E_d/K}}{\alpha_{E/F}} = \lim_{\substack{n \rightarrow \infty \\ n \text{ odd}}} \sqrt[n]{\frac{|E'(K_n)/NE'(F_n)|}{|E(K_n)/NE(F_n)|}}.$$

*If  $l \neq 2$ , this quotient is 1.*

(2) *We have  $\alpha_{E_d/K} = \alpha_{E/K}$  and  $\alpha_{E/F} = \alpha_{E/K}^2$  unless  $l = p$  and  $E$  has additive potentially supersingular reduction or  $l = p = 2$  and  $E$  has supersingular reduction.*

*Proof.* (1) We apply Lemma 4.6 for  $E/K_n, E/F_n$ ; because  $n$  is odd, we have  $F_n = K_n(\sqrt{d})$ . The minimal differentials stay the same in unramified extensions, so  $\alpha_{E/K_n} = \alpha_{E/K}^n$ , and similarly for  $\alpha_{E/F_n}$  and  $\alpha_{E_d/K_n}$ . Thus,

$$\frac{c_{E'/F_n}}{c_{E/F_n}} \frac{c_{E/K_n}}{c_{E'/K_n}} \frac{c_{E_d/K_n}}{c_{E'_d/K_n}} \frac{|E'(K_n)/NE'(F_n)|}{|E(K_n)/NE(F_n)|} = \frac{\alpha_{E/K}^n \alpha_{E_d/K}^n}{\alpha_{E/F}^n}.$$

All the Tamagawa numbers are bounded, so the claim follows by taking  $n$ th roots and letting  $n \rightarrow \infty$ . Moreover, if  $l \neq 2$ , because the norm quotients are 2-groups and  $\alpha$ 's are powers of  $l$ , we can compare the  $l$ -parts before taking the limit, and we find that  $\frac{\alpha_{E/K} \alpha_{E_d/K}}{\alpha_{E/F}} = 1$ .

(2) We may assume  $l = p$ , as otherwise all  $\alpha$ 's are 1 by Lemma 4.3.

(a) If  $l \neq 2$ , or  $E$  has either good ordinary or split multiplicative reduction, we have  $\alpha_{E/F} = \alpha_{E/K} \alpha_{E_d/K}$ : if  $l \neq 2$ , this is proved in (1); otherwise, the norm quotients have size at most 4 by [12] Prop 8.6, Prop 4.1 and  $\alpha_{E/F} = \alpha_{E/K} \alpha_{E_d/K}$  by (1).

(b) If either  $E$  is semistable or  $l = p > 3$  and  $E$  is potentially ordinary,

$$\frac{\phi^* \omega_{E'/K}}{\omega_{E/K}} = \frac{\phi^* \omega_{E'/F}}{\omega_{E/F}} \times \text{unit},$$

by Lemma 4.4, and

$$\alpha_{E/F} = \left| \frac{\phi^* \omega_{E'/F}}{\omega_{E/F}} \right|_F^{-1} = \left| \frac{\phi^* \omega_{E'/K}}{\omega_{E/K}} \right|_K^{-2} = \alpha_{E/K}^2.$$

(c) Combining (a) and (b), we find that  $\alpha_{E/K} = \alpha_{E_d/K}$  and  $\alpha_{E/F} = \alpha_{E/K}^2$  if either  $l > 3$  and  $E$  is semistable or potentially ordinary, or  $l = 3$  and  $E$  is semistable, or  $l = 2$  and  $E$  is split multiplicative or good ordinary. It follows that  $\alpha_{E/K} = \alpha_{E_d/K}$  and  $\alpha_{E/F} = \alpha_{E/K}^2$  also hold for quadratic twists of such curves, as  $\alpha_{E_{d_1}/K} = \alpha_{E/K} = \alpha_{E_{d_2}/K}$  for any pair of twists. Since a curve with potentially multiplicative reduction is a quadratic twist of a semistable one, and a curve with potentially ordinary reduction a quadratic twist of

a good ordinary one when  $l \leq 3$  (Theorem 3.2), the result holds in all the cases claimed.  $\square$

**Proposition 4.8.** *Suppose  $l = p$  and  $E$  has good ordinary reduction. Let  $F = K(\ker \phi)$  be the field obtained by adjoining the coordinates of points in  $\ker \phi$ . Then*

$$\frac{\phi^* \omega'}{\omega} = \begin{cases} p \cdot \text{unit}, & \text{if } \ker \phi \subset \hat{E}(\mathfrak{m}_F) \\ \text{unit}, & \text{otherwise} \end{cases}.$$

*Proof.* The isogeny  $\phi$  induces an isogeny on formal groups

$$\phi : \hat{E}(\mathfrak{m}_F) \rightarrow \hat{E}'(\mathfrak{m}_F), \quad \phi(T) = aT + \cdots,$$

with  $a = \frac{\phi^* \omega'}{\omega}$  by Lemma 4.2 (1). Define  $a'$  similarly for  $\phi^t$ . The reduction  $\tilde{E} = E \bmod \mathfrak{m}_F$  is an ordinary elliptic curve, so  $[p] = \tilde{\phi} \circ \tilde{\phi}^t$  is an isogeny of height 1 on its formal group. Hence either  $\tilde{\phi}$  or  $\tilde{\phi}^t$  is an isomorphism on formal groups of the reduced curves, in other words either  $a \bmod \mathfrak{m}_F$  or  $a' \bmod \mathfrak{m}_F$  is non-zero. Because  $aa' = p$ , one of  $a, a'$  is a unit and the other one is  $p$ -unit. If  $a$  is a unit, then  $\ker \phi$  is trivial on  $\hat{E}$ . Otherwise,  $\phi$  reduces to an inseparable isogeny of prime degree, and hence  $\ker \tilde{\phi} = 0$  on  $\tilde{E}$ . Therefore  $\ker \phi$  lies on the formal group.  $\square$

**Proposition 4.9.** *If  $l = p$  and  $E$  has potentially ordinary reduction, then  $\frac{\phi^* \omega'}{\omega}$  is either a unit or  $p$ -unit. If  $F/K$  is finite, then  $\frac{\phi^* \omega_{E'/F}}{\omega_{E/F}} = \frac{\phi^* \omega_{E'/K}}{\omega_{E/K}} \cdot \text{unit}$ .*

*Proof.* When  $p = 2$  or  $3$ , Theorem 3.2 shows that  $E$  is a quadratic twist of a curve with good reduction. The result follows from Propositions 4.8 and 4.7(2).

When  $p \geq 5$ , Lemma 4.4 shows that  $\frac{\phi^* \omega_{E'/K}}{\omega_{E/K}} = \frac{\phi^* \omega_{E'/F}}{\omega_{E/F}}$  for any  $F/K$ . Taking  $F$  to be the field where  $E$  acquires good reduction, we see that this quantity is a unit or  $p$ -unit by Proposition 4.8.  $\square$

**Proposition 4.10.** *If  $E$  has potentially multiplicative reduction, then*

$$\frac{\phi^* \omega'}{\omega} = \begin{cases} \text{unit}, & \text{if } v(j) = p v(j') \\ p \cdot \text{unit}, & \text{otherwise} \end{cases}.$$

*In particular,  $\frac{\phi^* \omega_{E'/K}}{\omega_{E/K}} = \frac{\phi^* \omega_{E'/F}}{\omega_{E/F}} \cdot \text{unit}$  for every finite extension  $F/K$ .*

*Proof.* Note that if  $l \neq p$ , the result follows from Lemma 4.3. Because both the  $j$ -invariant and  $\alpha$  are unchanged under quadratic twists (Proposition 4.7), we may assume that  $E$  has split multiplicative reduction.

By the theory of the Tate curve (Theorem 9.1), the pair  $E, E'$  is  $E^{(q^p)}, E^{(q)}$  (in some order), with  $q \in \mathfrak{m}_K$ . In particular, either  $v(j) = p v(j')$  or  $v(j') = p v(j)$ . Because

$$\frac{\phi^* \omega'}{\omega} \frac{(\phi^t)^* \omega}{\omega'} = \frac{\phi^* \omega'}{\omega} \frac{(\phi^* \phi^t)^* \omega}{\phi^* \omega'} = \frac{p \omega}{\omega} = p,$$

the claim for  $\phi$  is equivalent to that for  $\phi^t$ . Swapping  $E$  and  $E'$  if necessary, assume that  $E = E^{(q^p)}, E' = E^{(q)}$ , in which case  $\phi$  is given by

$$\phi : E(K) = K^\times / (q^p)^\mathbb{Z} \longrightarrow K^\times / q^\mathbb{Z} = E'(K),$$

induced by the identity map on  $K^\times$ . Here  $|\ker \phi| = p$ ,  $|\operatorname{coker} \phi| = 1$  on  $E(K)$ , and  $\frac{c}{c'} = \frac{v(q^p)}{v(q)} = p$ . By Lemma 4.2, the quotient  $\frac{\phi^* \omega'}{\omega}$  is a unit.  $\square$

## 5. DISCRIMINANTS AND KODAIRA TYPES II

### Theorem 5.1.

- (1) *If  $E$  has potentially good reduction, and either  $l \neq p$  or the reduction is good or potentially ordinary, then  $\delta = \delta'$ .*
- (2) *If  $E$  has multiplicative reduction, then  $\frac{\delta}{\delta'} = \frac{v(j)}{v(j')} = p^{\pm 1}$ .*
- (3) *If  $E$  has potentially multiplicative reduction, then*

$$\delta - \delta' = v(j') - v(j) = \begin{cases} \frac{1-p}{p}v(j), & \text{if } v(j) = pv(j') \\ (p-1)v(j), & \text{if } v(j') = pv(j) \end{cases}.$$

*Proof.* If  $E$  has good reduction, then  $\delta = \delta' = 0$ . If  $E$  has split multiplicative reduction, then  $E$  and  $E'$  are Tate curves with parameters  $q$  and  $q^p$ , in some order (Theorem 9.1). So  $\delta = -v(j)$  and  $\delta' = -v(j')$  are  $v(q)$  and  $pv(q)$ , in some order. Thus (1) holds in the good reduction case, and (2), (3) in the split multiplicative case.

If either  $l \neq p$  or  $E$  is potentially multiplicative, then  $\frac{\phi^* \omega_{E'/K}}{\omega_{E/K}} = \frac{\phi^* \omega_{E'/F}}{\omega_{E/F}}$  for any  $F/K$ , by Lemma 4.3 and Proposition 4.10. Taking  $F$  to be a field where  $E$  has good or split multiplicative reduction, we find that the claim for  $E/F$  implies that for  $E/K$ , by Lemma 4.4.

We are left with the case that  $E$  has additive potentially ordinary reduction with  $l = p$ . If  $p > 3$ , the claim is proved in Corollary 3.3. If  $p = 2, 3$ , then  $E/K$  is a quadratic twist of a curve  $E_d/K$  with good ordinary reduction (Theorem 3.2). Let  $F = K(\sqrt{d})$  be the corresponding quadratic extension. In the notation of Proposition 4.7 we have  $\alpha_{E/K} = \alpha_{E_d/K}$  and, since the minimal model of  $E_d$  stays minimal in  $F/K$  and  $E/F \cong E_d/F$ , also  $\alpha_{E/F} = \alpha_{E_d/F}^2$ . So  $\alpha_{E/F} = \alpha_{E/K}^2$ . In other words,  $\frac{\phi^* \omega_{E'/K}}{\omega_{E/K}} = \frac{\phi^* \omega_{E'/F}}{\omega_{E/F}}$ , and the claim follows, again by Lemma 4.4.  $\square$

**Remark 5.2.** Note that in the potentially good case, the formulae  $\delta = \delta'$  (Theorem 5.1) and  $\delta' \equiv p\delta \pmod{12}$  (Theorem 3.1) do not contradict each other. The reason is that the possible reduction types are restricted in the potentially ordinary case, see Theorem 3.2.

**Remark 5.3.** In the potentially supersingular case the formulae in Proposition 4.9 and Theorem 5.1(1) may not hold. For example, consider the 5-isogenous elliptic curves  $E = 50b1$ ,  $E' = 50b3$ . Their reduction types over  $\mathbb{Q}_5$  are II and II\* respectively, so  $\delta \neq \delta'$ . Also, over  $\mathbb{Q}_5(\sqrt{5})$  the reduction types become IV and IV\*, and  $\phi^* \omega' / \omega = \sqrt{5}$  (computed as in Lemma 4.4 from the minimal discriminants), which is neither a unit nor 5 times a unit.

**Theorem 5.4.**

- (1) If  $E$  has potentially good reduction and  $p \neq l$ , then the Kodaira types of  $E$  and  $E'$  are the same.
- (2) If  $E$  has potentially good ordinary reduction and  $p = l$ , then the Kodaira types of  $E$  and  $E'$  are the same.
- (3) If  $E$  has tame potentially good supersingular reduction and  $p = l$ , then  $E$  and  $E'$  have opposite Kodaira type ( $\text{II} \leftrightarrow \text{II}^*$ ,  $\text{III} \leftrightarrow \text{III}^*$ ,  $\text{IV} \leftrightarrow \text{IV}^*$ ,  $\text{I}_0^* \leftrightarrow \text{I}_0^*$ ).
- (4) If  $E$  has multiplicative reduction, then the Kodaira type is  $\text{I}_n$  for  $E$ , and either  $\text{I}_{pn}$  or  $\text{I}_{n/p}$  for  $E'$ , corresponding to  $v(j') = pv(j)$  and  $pv(j') = v(j)$ .
- (5) If  $E$  has additive potentially multiplicative reduction, then the Kodaira type is  $\text{I}_n^*$  for  $E$  and  $\text{I}_{n'}^*$  for  $E'$ , where either

$$v(j') = pv(j) \quad \text{and} \quad n' = pn - 4(a-1)(p-1) = n - v(j)(p-1)$$

or vice versa (swap  $n \leftrightarrow n'$ ,  $j \leftrightarrow j'$ ); here  $a$  is the conductor exponent of the quadratic character  $K(\sqrt{-c_6})/K$  (it is 1 if  $l \neq 2$ ), where  $c_6$  is the standard invariant of  $E$  as in [19] §III.1.

*Proof.* Write  $f, f'$  for the conductor exponents of  $E$  and  $E'$ , and  $m, m'$  for the number of connected components of the special fibre of their Néron models. Because  $\phi$  induces an isomorphism between the  $\ell$ -adic Tate modules of  $E$  and  $E'$  for  $\ell \neq l, p$ , we have  $f = f'$ . Recall that  $\delta = f + m - 1$  and  $\delta' = f' + m' - 1$  by Ogg's formula [20] IV.11.1.

(1) By Theorem 5.1,  $\delta = \delta'$  and so  $m = m'$ . If  $l \neq 2$ , then from the reduction type table [20] IV.9, Table 4.1 we see that the Kodaira type in the additive potentially good case is determined by  $m$ , so they are the same for  $E$  and  $E'$ . (Note that  $\text{I}_n^*$  is necessarily potentially multiplicative even when  $l = 3$ .)

Suppose  $l = 2$ . Then  $m$  almost determines the reduction type, except for the pairs  $(\text{I}_2^*, \text{IV}^*)$ ,  $(\text{I}_3^*, \text{III}^*)$  and  $(\text{I}_4^*, \text{II}^*)$ . Passing to the maximal unramified extension if necessary, we see that  $\text{I}_0^*$  and  $\text{I}_n^*$  are the only reduction types with (2-part of) the local Tamagawa number equal to 4. Since the 2-part of the Tamagawa number is invariant under  $\phi$  (as  $p = \deg \phi$  is odd,  $\phi$  induces an isomorphism between the 2-parts of  $E/E_0$  and  $E'/E'_0$ ), the Kodaira types must be the same.

(2) In the tame case, this is Corollary 3.3. By Theorem 3.2, the only wild case is when  $p = l = 2$  and  $E, E'$  are quadratic twists of curves with good ordinary reduction by some character  $\chi$ . Here  $\delta = \delta'$  by Theorem 5.1(1), so  $m = m'$  as in (1). Again as in (1), the Kodaira types of  $E$  and  $E'$  are the same, except possibly for 3 pairs of cases  $(\text{I}_2^*, \text{IV}^*)$ ,  $(\text{I}_3^*, \text{III}^*)$  and  $(\text{I}_4^*, \text{II}^*)$ . We claim that none of these can occur. For the first one, the reduction type  $\text{IV}^*$  is tame by Theorem 3.1. For the second one,  $m = 8$  and  $6|\delta$  (since  $E$  acquires good reduction after a quadratic extension), so  $f$  is odd by Ogg's formula; however  $f$  is twice the conductor exponent of  $\chi$ , contradiction. In

the last case, pass to the maximal unramified extension as in (1). Then the Tamagawa numbers of  $E$  and  $E'$  become 1 and 4 ([20] IV.9, Table 4.1), but their quotient is 1, 2 or  $\frac{1}{2}$  by the very last case of Theorem 6.1. (The proof of this case does not use the present theorem.)

(3) This is a special case of Corollary 3.3.

(4) This follows from Theorem 9.1.

(5) The quadratic twists  $E_{-c_6}, E'_{-c_6}$  have split multiplicative reduction and are  $p$ -isogenous (Lemma 9.2, Remark 4.5). If  $v(j') = pv(j)$ , then these twists have Kodaira types  $I_\nu, I_{p\nu}$  with  $-\nu = v(j_{E_{-c_6}}) = v(j)$  (Theorem 9.1). By Theorem 9.5,  $E$  and  $E'$  have Kodaira types  $I_n^*$  and  $I_{n'}^*$  with  $n = \nu + 4a - 4$  and  $n' = p\nu + 4a - 4$ . Clearly

$$n' = p(n - 4a + 4) + 4a - 4 = pn - 4(p - 1)(a - 1).$$

Because  $v(j) = -\nu = -n + 4a - 4$ , also

$$n' = pn - (p - 1)(4a - 4) = pn - (p - 1)(v(j) + n) = n - (p - 1)v(j).$$

If, on the other hand,  $v(j) = pv(j')$ , swap  $E$  and  $E'$ .  $\square$

## 6. TAMAGAWA NUMBERS

**Theorem 6.1.** *If  $E$  is semistable, then the ratio of Tamagawa numbers  $\frac{c}{c'}$  is*

- 1 if  $E$  has good reduction.
- $\frac{\delta}{\delta'} = \frac{v_K(j(E))}{v_K(j(E'))} = p^{\pm 1}$  if  $E$  has split multiplicative reduction.
- if  $E$  has nonsplit multiplicative reduction:
  - 1 if  $p \neq 2$ , or if both  $\delta$  and  $\delta'$  are even,
  - 2 if  $p = 2$  and  $\delta'$  is odd,
  - $\frac{1}{2}$  if  $p = 2$  and  $\delta$  is odd.

If  $E$  has additive reduction and  $p > 3$ , then  $c/c' = 1$ .

If  $E$  has additive reduction and  $p = 3$ , then  $c/c'$  is

- 1 if  $l \neq 3$ , unless  $E$  has type IV, IV\* and  $\mu_3 \not\subset K$ . In this exceptional case,
  - 3 if  $E(K)[3] \neq 0$ ,
  - $\frac{1}{3}$  if  $E(K)[3] = 0$ .
- 1 if  $l = 3$  and  $E$  has Kodaira type III, III\*, I<sub>0</sub>\*, I<sub>n</sub>\* (equivalently,  $E$  does not have wild potentially supersingular reduction).

If  $E$  has additive reduction and  $p = 2$ , then  $c/c'$  is

- 1 if  $l \neq 2$  and  $E$  is not of type I<sub>0</sub>\* or I<sub>n</sub>\*.
- 1 if  $l = 2$  and  $E$  has tame potentially good reduction (i.e. type IV, IV\*).
- if  $l \neq 2$  and  $E$  has type I<sub>0</sub>\* or I<sub>n</sub>\*, or  $l = 2$  and  $E$  does not have potentially supersingular reduction,
  - 1 if  $\frac{\Delta}{\Delta'}$  is a norm in  $F/K$ ,
  - $\frac{1}{2}$  if  $\Delta'$  is a norm in  $F/K$  and  $\Delta$  is not,
  - 2 if  $\Delta$  is a norm in  $F/K$  and  $\Delta'$  is not,

where  $F/K$  is a quadratic extension such that  $E/F$  has good or split multiplicative reduction.

**Lemma 6.2.** *The quotient  $c/c'$  is a power of  $p$ .*

*Proof.* The isogeny  $\phi$  induces maps  $E(K) \rightarrow E'(K)$  and  $E_0(K) \rightarrow E'_0(K)$ , and so  $E/E_0 \rightarrow E'/E'_0$ . These are finite groups, and since  $\phi\phi^t = [p] = \phi\phi^t$  are automorphisms on their prime-to- $p$  parts,  $\phi$  is an isomorphism between these prime-to- $p$  parts.  $\square$

*Proof of Theorem 6.1.* In the semistable case this follows from Tate's algorithm [20] IV.9 and Theorem 5.1. Assume henceforth that  $E$  and  $E'$  have additive reduction. In particular  $1 \leq c, c' \leq 4$  (cf. [19] VII.6.1, [20] IV.9 Table 4.1), so for  $p > 3$  the result follows by Lemma 6.2.

For  $p = 3$ ,  $l \neq 3$  see [5] Lemma 11.

Suppose  $p = 2$ ,  $l \neq 2$ . If the Kodaira type is not  $I_0^*$  or  $I_n^*$ , the Kodaira types of  $E$  and  $E'$  are the same by Theorem 5.4. By the reduction type table ([20] IV.9, Table 4.1) in the case II, II\*, IV, IV\* the 2-parts of the Tamagawa numbers are trivial, and in the case III, III\* they are both 2. Hence  $c$  and  $c'$  have the same 2-part, and are therefore equal. When the reduction type is  $I_0^*$ , see the computation in [5] §7.4.

For  $p = l = 3$  and type III, III\*,  $I_0^*$ , the isogenous curve  $E'$  also has one of these three Kodaira types (Theorem 3.1). The Tamagawa numbers for these types can be 1, 2 or 4, so the 3-isogeny forces the equality  $c = c'$  (Lemma 6.2). Similarly, for  $p = l = 2$  and type IV, IV\*, the Tamagawa numbers are 1 or 3, and are unchanged by a 2-isogeny.

Finally, in the three remaining cases ( $p = l = 3$ , type  $I_n^*$ ;  $p = l = 2$ ,  $E$  not potentially supersingular; or  $p = 2$ ,  $l \neq 2$ , type  $I_n^*$ ), some quadratic twist  $E_d/K$  of  $E/K$  has either good or split multiplicative reduction (Theorem 3.2, Lemma 9.2). Let  $F = K(\sqrt{d})$ , the field over which  $E$  becomes semistable. By Lemma 4.6,

$$\left( \alpha_{E/K} \frac{c_{E'/K}}{c_{E/K}} \right)^{-1} \cdot \alpha_{E/F} \frac{c_{E'_d/F}}{c_{E_d/F}} \cdot \left( \alpha_{E_d/K} \frac{c_{E'_d/K}}{c_{E_d/K}} \right)^{-1} \cdot \frac{\left| \frac{E'_d(K)}{NE'_d(F)} \right|}{\left| \frac{E_d(K)}{NE_d(F)} \right|} = 1,$$

where  $\alpha_{E/K} = \left| \frac{\phi^* \omega'}{\omega} \right|_K^{-1}$  and  $\alpha_{E_d/K}$  and  $\alpha_{E/F}$  are similarly defined for  $E_d/K$  and  $E_d/F \cong E/F$ . Proposition 4.7(2) shows that  $\alpha_{E/K} = \alpha_{E_d/K}$ , and  $\alpha_{E/F} = \alpha_{E_d/F}^2 = \alpha_{E/K} \alpha_{E_d/K}$ . Also,

$$\frac{c_{E'_d/F}}{c_{E_d/F}} = \frac{c_{E'_d/K}}{c_{E_d/K}}$$

by the good and split multiplicative cases of the theorem. Hence

$$\frac{c_{E'/K}}{c_{E/K}} = \left| \frac{E'_d(K)}{NE'_d(F)} \right| / \left| \frac{E_d(K)}{NE_d(F)} \right|.$$

If  $p = 3$ , then  $\frac{c_{E'/K}}{c_{E/K}}$  is a power of 3 (Lemma 6.2) but the groups in the right-hand side are 2-groups, so  $c_{E'/K} = c_{E/K}$ .

Finally, suppose  $p = 2$ . If  $E_d$  has good reduction (so  $l = p$  and the reduction is good ordinary), by [12] Prop 8.6  $\frac{E_d(K)}{NE_d(F)}$  has order 4 or 2, corresponding to whether  $\Delta_{E_d/K}$  is a norm in  $F/K$  or not, and similarly for  $E'$ . This gives the result for  $c/c'$ , noting that  $\Delta_{E_d/K}$  can be replaced by  $\Delta_{E/K}$  in this criterion, since they differ by a 6th power (Lemma 9.4).

If  $E_d$  has split multiplicative reduction, by [12] Prop 4.1  $\frac{E_d(K)}{NE_d(F)}$  has order 2 or 1, depending on whether the Tate parameter  $q$  of  $E_d/K$  is a norm in  $F/K$  or not. Because  $\Delta_{E_d/K}/q = \prod (1 - q^n)^{24}$  is a square ([20] §V.3) and  $\Delta_{E/K}/\Delta_{E_d/K}$  is a 6th power, we get the same result as in the potentially ordinary case.  $\square$

## 7. REAL AND COMPLEX PERIODS

**Notation 7.1.** In this section the field  $\mathcal{K}$  will be  $\mathbb{R}$  or  $\mathbb{C}$ , and  $\phi : E \rightarrow E'$  a  $\mathcal{K}$ -rational  $p$ -isogeny of elliptic curves over  $\mathcal{K}$ .

**Definition 7.2.** The *period* of an elliptic curve  $E/\mathcal{K}$  with respect to an invariant differential  $\omega$  is

$$\Omega(E, \omega) = \int_{E(\mathcal{K})} |\omega| \quad \text{if } \mathcal{K} \cong \mathbb{R},$$

and

$$\Omega(E, \omega) = 2 \int_{E(\mathcal{K})} \omega \wedge \bar{\omega} \quad \text{if } \mathcal{K} \cong \mathbb{C}.$$

**Remark 7.3.** For  $\mathcal{K} = \mathbb{R}$ , one sometimes uses the period  $\Omega^+$ , which is obtained by integrating only over the connected component of  $E(\mathbb{R})$ . Thus  $\Omega(E, \omega) = \Omega^+$  or  $2\Omega^+$ , depending on whether or not  $E(\mathbb{R})$  is connected.

**Lemma 7.4.** *The periods of  $E$  and  $E'$  satisfy*

$$\frac{\Omega(E, \omega)}{\Omega(E', \omega')} = \frac{|\ker \phi : E(\mathcal{K}) \rightarrow E'(\mathcal{K})|}{|\operatorname{coker} \phi : E(\mathcal{K}) \rightarrow E'(\mathcal{K})|} \cdot \left| \frac{\omega}{\phi^* \omega'} \right|_{\mathcal{K}}.$$

*Proof.*

$$\frac{\Omega(E, \omega)}{\Omega(E', \omega')} = \frac{\Omega(E, \phi^* \omega')}{\Omega(E', \omega')} \cdot \frac{\Omega(E, \omega)}{\Omega(E, \phi^* \omega')} = \frac{|\ker \phi : E(\mathcal{K}) \rightarrow E'(\mathcal{K})|}{|\operatorname{coker} \phi : E(\mathcal{K}) \rightarrow E'(\mathcal{K})|} \cdot \left| \frac{\omega}{\phi^* \omega'} \right|_{\mathcal{K}}.$$

$\square$

**Remark 7.5.** If  $K$  is an  $l$ -adic field with residue field  $k$ , then

$$\int_{E(K)} |\omega|_K = c_{E/K} \frac{|E(k)|}{|k|},$$

and  $\frac{|E(k)|}{|k|}$  is the value of the Euler factor of  $E$  at  $s = 1$ . These ‘local integrals’ enter Tate’s formulation of the Birch–Swinnerton-Dyer conjecture [21]. Lemma 7.4 is the Archimedean analogue of Lemma 4.2(2).

**Proposition 7.6.** *The quotient  $\frac{|\ker \phi: E(\mathcal{K}) \rightarrow E'(\mathcal{K})|}{|\operatorname{coker} \phi: E(\mathcal{K}) \rightarrow E'(\mathcal{K})|}$  is*

- $p$  if  $\mathcal{K} = \mathbb{C}$ ,
- $p$  if  $\mathcal{K} = \mathbb{R}$ ,  $p \neq 2$  and  $\ker \phi \subset E(\mathcal{K})$ ,
- $1$  if  $\mathcal{K} = \mathbb{R}$ ,  $p \neq 2$  and  $\ker \phi \not\subset E(\mathcal{K})$ .

*If  $\mathcal{K} = \mathbb{R}$  and  $p = 2$ , write  $E$  in the form  $y^2 = x^3 + ax^2 + bx$  so that  $(0, 0) \in \ker \phi$ . In this case, the quotient is*

- $1$  if  $b > 0$ , and either  $a < 0$  or  $4b > a^2$ ,
- $2$  otherwise.

*Proof.* When  $p \neq 2$ , the cokernel is trivial and the result follows immediately. For  $p = 2$ , the kernel has size 2, and the cokernel is computed in [5] §7.1.  $\square$

## 8. PERIODS OF ELLIPTIC CURVES OVER $\mathbb{Q}$

Finally, we turn to periods of elliptic curves over  $\mathbb{Q}$ .

**Notation 8.1.** For an elliptic curve  $E$  over  $\mathbb{Q}$  we write  $\omega$  for the global minimal differential on  $E$  and

$$\Omega = \Omega(E/\mathbb{R}, \omega), \quad \Omega_{\mathbb{C}} = \Omega(E/\mathbb{C}, \omega)$$

for its real and complex periods. We similarly use  $\omega', \Omega'$  and  $\Omega'_{\mathbb{C}}$  for  $E'/\mathbb{Q}$ .

**Theorem 8.2.** *Let  $\phi: E \rightarrow E'$  be a rational isogeny of elliptic curves over  $\mathbb{Q}$ . Then the quotient  $\Omega/\Omega'$  is  $p, 1$  or  $p^{-1}$ , and the following are equivalent:*

- (1)  $\Omega = \Omega'$ .
- (2)  $\sum_l \operatorname{ord}_p\left(\frac{c_{E/\mathbb{Q}_l}}{c_{E'/\mathbb{Q}_l}}\right) \equiv \operatorname{ord}_{s=1} L(E, s) \pmod{2}$ .

*If  $p \neq 2, 3$ , this is also equivalent to*

- (3)  $E$  has an odd number of primes of additive reduction with local root number  $-1$ .

*The quotient  $\Omega_{\mathbb{C}}/\Omega'_{\mathbb{C}}$  is  $p$  or  $p^{-1}$ , and it is  $p$  if and only if  $\omega = \pm \phi^* \omega'$ .*

*Proof.* By Lemma 7.4,

$$\frac{\Omega}{\Omega'} = \frac{|\ker \phi: E(\mathbb{R}) \rightarrow E'(\mathbb{R})|}{|\operatorname{coker} \phi: E(\mathbb{R}) \rightarrow E'(\mathbb{R})|} \cdot \left| \frac{\omega}{\phi^* \omega'} \right|.$$

The first term is  $p$  or  $1$  by Proposition 7.6, and the second term  $\left| \frac{\omega}{\phi^* \omega'} \right|$  is either  $1$  or  $p^{-1}$ . So  $\Omega/\Omega' \in \{1, p, p^{-1}\}$ . By the same argument over  $\mathbb{C}$ ,  $\left| \frac{\omega}{\phi^* \omega'} \right|_{\mathbb{C}}$  is either  $1$  or  $p^{-2}$ , which immediately gives the claim for the complex periods.

It remains to prove the equivalence of (1), (2) and (3). Now  $\Omega = \Omega'$  if and only if  $\operatorname{ord}_p\left(\frac{\Omega}{\Omega'}\right)$  is even, and we can relate this to the parity of the analytic rank and of the  $p^\infty$ -Selmer rank of  $E/\mathbb{Q}$ :

$$\operatorname{ord}_{s=1} L(E, s) \equiv \operatorname{rk}_p E/\mathbb{Q} \equiv \operatorname{ord}_p \frac{\Omega}{\Omega'} + \sum_l \operatorname{ord}_p\left(\frac{c_{E/\mathbb{Q}_l}}{c_{E'/\mathbb{Q}_l}}\right) \pmod{2},$$



by the  $p$ -parity conjecture over  $\mathbb{Q}$  ([6] Thm. 1.4) and Cassels' formula for the parity of the  $p^\infty$ -Selmer rank for an elliptic curve with a  $p$ -isogeny ([6] Rmk. 4.4). This proves the equivalence (1)  $\Leftrightarrow$  (2).

For (2)  $\Leftrightarrow$  (3), suppose  $p \geq 5$ . Then by Theorem 6.1,  $\text{ord}_p(\frac{c_{E/\mathbb{Q}_l}}{c_{E'/\mathbb{Q}_l}})$  is odd if and only if  $E$  has split multiplicative reduction at  $l$ . So the left-hand side in (2) is the number of primes of split multiplicative reduction. The right-hand side is determined by the global root number  $w$  of  $E$ : it is even if  $w = +1$  and odd if  $w = -1$ . Because  $w$  is the product of local root numbers,

$$w = - \prod_l w_l$$

and the local root numbers  $w_l = \pm 1$  are  $-1$  for primes of split multiplicative reduction and  $+1$  for primes of good and nonsplit multiplicative reduction, the result follows.  $\square$

**Lemma 8.3.** *Suppose  $K/\mathbb{Q}_p$  is unramified. There are no elliptic curves over  $K$  with good supersingular reduction that admit a  $p$ -isogeny.*

*Proof.* If  $E$  has good supersingular reduction and  $K/\mathbb{Q}_p$  is unramified, Serre [17] Prop. 12 proves that the image of Galois in  $\text{Aut } E[p]$  is the normaliser of the non-split Cartan subgroup of  $\text{GL}_2(\mathbb{F}_p)$ . In particular, it acts irreducibly on  $E[p]$ , so  $E/K$  cannot have a  $p$ -isogeny.  $\square$

**Lemma 8.4.** *Suppose  $K/\mathbb{Q}_p$  is unramified,  $p$  is odd,  $E/K$  is semistable and  $\omega, \omega'$  are minimal differentials on  $E$  and  $E'$ . If  $\ker \phi \subset E(K)$ , then  $\frac{\omega}{\phi^* \omega'}$  is a  $p$ -adic unit.*

*Proof.* The curve  $E$  cannot have supersingular reduction (Lemma 8.3) or non-split multiplicative reduction (as  $\ker \phi \cong \mathbb{Z}/p\mathbb{Z}$  or  $\mu_p$  in the split multiplicative case, and it has no points after an unramified quadratic twist). In the good ordinary case, Proposition 4.8 gives the claim. In the split multiplicative case, see the proof of Proposition 4.10.  $\square$

**Lemma 8.5.** *If  $\phi : E \rightarrow E'$  is a  $p$ -isogeny of elliptic curves over  $\mathbb{Q}$ , with  $p$  odd,  $\ker \phi \subset E(\mathbb{Q})$  and  $E$  is semistable at  $p$ , then  $\frac{\Omega}{\Omega'} = \frac{\Omega_{\mathbb{C}}}{\Omega'_{\mathbb{C}}} = p$ .*

*Proof.* We have

$$\frac{\Omega}{\Omega'} \stackrel{7.4}{=} \frac{|\ker \phi : E(\mathbb{R}) \rightarrow E'(\mathbb{R})|}{|\text{coker } \phi : E(\mathbb{R}) \rightarrow E'(\mathbb{R})|} \cdot \left| \frac{\omega}{\phi^* \omega'} \right| \stackrel{7.6}{=} p \left| \frac{\omega}{\phi^* \omega'} \right| \stackrel{8.4}{=} p,$$

and similarly for  $\Omega_{\mathbb{C}}$ .  $\square$

**Remark 8.6.** Note from the proof of the lemma that without the semistability assumption for real periods we still have  $\Omega \geq \Omega'$  when  $\ker \phi \subset E(\mathbb{Q})$ .

**Theorem 8.7.** *Let  $\phi : E \rightarrow E'$  be a  $p$ -isogeny of semistable elliptic curves over  $\mathbb{Q}$ , with  $p$  odd. Then  $\frac{\Omega}{\Omega'} = \frac{\Omega_{\mathbb{C}}}{\Omega'_{\mathbb{C}}}$  is either  $p$  or  $p^{-1}$ . Moreover,*

$$\frac{\Omega}{\Omega'} = \frac{\Omega_{\mathbb{C}}}{\Omega'_{\mathbb{C}}} = p \iff \frac{\omega}{\phi^* \omega'} = \pm 1 \iff \ker \phi \subset E(\mathbb{Q}).$$

*Proof.* If  $\ker \phi \subset E(\mathbb{Q})$ , then  $\frac{\omega}{\phi^* \omega'}$  is a  $p$ -adic unit by Lemma 8.4 and unit at

all other primes by Lemma 4.3, so it is  $\pm 1$ . Also,  $\frac{\Omega}{\Omega'} = \frac{\Omega_{\mathbb{C}}}{\Omega'_{\mathbb{C}}} = p$  by Lemma 8.5. If  $\ker \phi \not\subset E(\mathbb{Q})$ , then by a result of Serre ([17] p. 307),  $\ker \phi^t \subset E'(\mathbb{Q})$ . The result now follows from that for  $\phi^t$ , as

$$p \cdot \frac{\omega}{\phi^* \omega'} = \frac{\phi^*(\phi^t)^* \omega}{\phi^* \omega'} = \frac{(\phi^t)^* \omega}{\omega'} = \pm 1.$$

□

## 9. APPENDIX: TATE CURVE AND QUADRATIC TWISTS

For completeness, we recall the following well-known facts. As usual,  $K$  is a finite extension of  $\mathbb{Q}_l$ , and the notation is as in §1.1.

**Theorem 9.1.** *An elliptic curve  $E/K$  with split multiplicative reduction of type  $I_n$  is isomorphic to a Tate curve  $E^{(q)}/K$  for some Tate parameter  $q \in \mathfrak{m}_K$  with  $v(q) = n = \delta = -v(j) = c$ . For a prime  $p$ ,*

$$E^{(q)}(\bar{K}) \cong \bar{K}^\times / q^{\mathbb{Z}} \quad \text{and} \quad E^{(q)}[p] \cong \langle \zeta_p, \sqrt[p]{q} \rangle$$

as Galois modules. There is a  $K$ -rational  $p$ -isogeny

$$\begin{array}{ccccc} E^{(q)}(K) & = & K^\times / q^{\mathbb{Z}} & \longrightarrow & K^\times / q^{p\mathbb{Z}} = E^{(q^p)}(K) \\ & & z & \mapsto & z^p. \end{array}$$

The other  $p$ -isogenies from  $E^{(q)}$  are parametrised by choices of a  $p$ th root of  $q$  in  $\bar{K}^\times$ , and given by

$$\begin{array}{ccccc} E^{(q)}(K) & = & K^\times / q^{\mathbb{Z}} & \longrightarrow & K^\times / (\sqrt[p]{q})^{\mathbb{Z}} = E^{(\sqrt[p]{q})}(K) \\ & & z & \mapsto & z. \end{array}$$

Such an isogeny is defined over  $K$  if and only if  $\sqrt[p]{q} \in K$ .

*Proof.* For the basic theory of the Tate curve, see [20] §V.3-V.5. For the statements about isogenies, see [16] §A.1.4 (Theorem and the proof of (2)  $\implies$  (1)), and the description of the function field of  $E_q$  in §A.1.1. □

**Lemma 9.2.** *If  $E/K$  has potentially multiplicative reduction, both the quadratic twist of  $E$  by  $-c_6$  and  $E/K(\sqrt{-c_6})$  have split multiplicative reduction. Here  $c_6$  is the standard invariant of  $E$  as in [19] §III.1.*

*Proof.* See [20] §V.5. □

**Lemma 9.3.** *Let  $E/K : y^2 = f(x)$  be an elliptic curve with additive potentially good reduction. Then the following are equivalent:*

- (1)  *$E$  has good reduction over a quadratic extension  $K(\sqrt{d})$ .*
- (2) *The quadratic twist  $E_d/K : dy^2 = f(x)$  has good reduction.*
- (3) *The inertia group  $\text{Gal}(\bar{K}/K^{nr})$  acts on the  $\ell$ -adic Tate module of  $E$  ( $\ell \neq l$ ) through  $\text{Gal}(K^{nr}(\sqrt{d})/K^{nr})$ .*

*Proof.* This follows from the criterion of Néron-Ogg-Shafarevich, and the fact that the Tate module of  $E_d$  is the Tate module of  $E$  twisted by the character of  $\text{Gal}(K(\sqrt{d})/K)$  of order 2. □

**Lemma 9.4.** *Let  $E/K$  be an elliptic curve, and  $E_d/K$  its quadratic twist by  $d \in K^\times$ . Then the minimal discriminants of  $E$  and  $E_d$  are related by  $\Delta_{E/K} = d^6 b^{12} \Delta_{E_d/K}$  for some  $b \in K^\times$ .*

*Proof.* Choose models  $E : y^2 = x^3 + ax + b$  and  $E_d : y^2 = x^3 + d^2 ax + d^3 b$ . Their discriminants differ by  $d^6$ , and they differ from the minimal discriminants by 12th powers.  $\square$

**Theorem 9.5** ([14] Thm 2.8). *Suppose  $E/K$  has multiplicative reduction of type  $I_n$ , so  $n = \delta = -v(j)$ . Let  $K(\sqrt{d})/K$  be a quadratic extension, and  $\chi : \text{Gal}(K(\sqrt{d})/K) \rightarrow \pm 1$  the corresponding character. Then the quadratic twist  $E_d$  has potentially multiplicative reduction of type  $I_{n+4f_\chi-4}^*$ , where  $f_\chi$  is the conductor exponent of  $\chi$ . If  $p \neq 2$ , then  $f_\chi = 1$  and the type is  $I_n^*$ .*

## 10. APPENDIX: VALUES OF MODULAR FORMS

We briefly recall the precise connection between values of modular forms on  $\Gamma_0(N)$  and invariants of elliptic curves with a cyclic  $N$ -isogeny. The modern approach of Katz is described in [9]. We give a low-tech description, which relies only on classical results.

Let  $E/\mathbb{C}$  be an elliptic curve with an invariant differential  $\omega$ . Put  $(E, \omega)$  in the form

$$(10.1) \quad E : y^2 = 4x^3 + ax + b, \quad \omega = \frac{dx}{y}.$$

By the uniformisation theorem, there is a unique lattice  $\Lambda = \mathbb{Z}\Omega_1 + \mathbb{Z}\Omega_2 \subset \mathbb{C}$  such that

$$\begin{aligned} \phi : \mathbb{C}/\Lambda &\longrightarrow E(\mathbb{C}) \\ z &\longmapsto (\wp_\Lambda(z), \wp'_\Lambda(z)) \end{aligned}$$

is an isomorphism (of complex Lie groups), and  $\phi^* \frac{dx}{y} = dz$ . Here  $\wp_\Lambda(z)$  is the Weierstrass  $\wp$ -function

$$\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{v \in \Lambda \setminus \{0\}} \left( \frac{1}{(z-v)^2} - \frac{1}{v^2} \right).$$

The coefficients of  $E$  are  $a = -60G_4(\Lambda)$  and  $b = -140G_6(\Lambda)$ , where the  $G_k$  are the standard modular functions

$$G_k(\Lambda) = \sum_{v \in \Lambda \setminus \{0\}} v^{-k}.$$

Let  $\tau = \frac{\Omega_2}{\Omega_1}$ , changing sign of  $\Omega_2$  if necessary to get  $\tau \in \mathbb{H}$ . Write  $\Lambda_\tau = \mathbb{Z}\tau + \mathbb{Z}$  and  $q = e^{2\pi i\tau}$ . Then

$$\Lambda = \Omega_1 \Lambda_\tau, \quad G_k(\Lambda) = \Omega_1^{-k} G_k(\Lambda_\tau),$$

and  $\tau \mapsto G_k(\Lambda_\tau)$  is, up to a constant, the Eisenstein series of weight  $k$ ,

$$E_k(\tau) = \frac{1}{2\zeta(k)} G_k(\Lambda_\tau) = 1 + \frac{2}{\zeta(1-2k)} \sum_{n=1}^{\infty} \frac{n^{2k-1} q^n}{1-q^n}.$$

The modular discriminant function  $\Delta(\tau) = \frac{1}{1728}(E_4(\tau)^3 - E_6(\tau)^2)$  satisfies

$$(10.2) \quad \Delta(\tau) = -16 \left( 4\left(\frac{a}{4}\right)^3 + 27\left(\frac{b}{4}\right)^2 \right) \cdot \left( \frac{\Omega_1}{2\pi} \right)^{12} = \left( \frac{\Omega_1}{2\pi} \right)^{12} \Delta_E,$$

where  $\Delta_E$  is the discriminant of the Weierstrass model  $y^2 = x^3 + \frac{a}{4}x + \frac{b}{4}$  of  $E$  obtained by rescaling  $y \mapsto 2y$  in (10.1) (so  $\frac{dx}{y}$  becomes  $\frac{dx}{2y}$ ).

Now suppose that the pair  $(E, \omega)$  is defined over a subfield  $\mathcal{K} \subset \mathbb{C}$ . Then  $a, b, \Delta \in \mathcal{K}$ , and therefore

$$\left( \frac{2\pi}{\Omega_1} \right)^4 E_4(\tau), \quad \left( \frac{2\pi}{\Omega_1} \right)^6 E_6(\tau), \quad \left( \frac{2\pi}{\Omega_1} \right)^{12} \Delta(\tau) \in \mathcal{K}.$$

In fact, suppose  $f(\tau)$  is any modular form on  $\Gamma_0(N)$  whose  $q$ -expansion has  $\mathcal{K}$ -rational coefficients. For any choice of non-negative integers  $m, n, o$  with  $4m + 6n + k = 12o$ , the form  $\tilde{f} = f E_4^m E_6^n / \Delta^o$  on  $\Gamma_0(N)$  has weight 0 (i.e. is a modular function). By a classical theorem (see [4] Thm. 11.9(b)),

$$\tilde{f}(\tau) = F(j(\tau), j(N\tau))$$

for some rational function  $F \in \mathbb{C}(x, y)$ . In fact,  $F \in \mathcal{K}(x, y)$  since  $F$  has a  $\mathcal{K}$ -rational  $q$ -expansion.<sup>1</sup> Summarising the whole discussion, we have

**Theorem 10.3.** *Let  $f \in M_k(\Gamma_0(N))$  be a modular form whose  $q$ -expansion has  $\mathcal{K}$ -rational coefficients,  $\mathcal{K} \subset \mathbb{C}$ . There are natural numbers  $m, n, o$  and a rational function  $F \in \mathcal{K}(x, y)$  such that for every cyclic isogeny of elliptic curves  $\phi : E \rightarrow E'$  of degree  $N$ , with  $E$  in the form (10.1) with corresponding complex lattices  $\Lambda = \mathbb{Z}\Omega_1 + \mathbb{Z}\Omega_2 \subset \mathbb{C}$ ,  $\Lambda' = \mathbb{Z}\Omega_1 + \mathbb{Z}N\Omega_2 \subset \mathbb{C}$ , we have*

$$\left( \frac{2\pi}{\Omega_1} \right)^k f\left( \frac{\Omega_2}{\Omega_1} \right) = \frac{a^m b^n}{\Delta_E^o} F(j(E), j(E')).$$

*In particular, if  $E$  and  $E'$  are defined over  $\mathcal{K}$ , the left-hand side lies in  $\mathcal{K}$ .*

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<sup>1</sup>This is clear for rational functions of  $j(\tau)$ , since the  $q$ -expansion of  $j(\tau)$  is rational, hence Galois invariant, and  $\mathbb{C}(t)^{\text{Aut}(\mathbb{C}/\mathcal{K})} = \mathcal{K}(t)$ ; in general, write  $\tilde{f}$  as a unique polynomial in  $j(N\tau)$  with coefficients in  $\mathbb{C}(j(\tau))$  of degree  $< n$ , where  $n$  is the degree of the modular polynomial  $\Phi_N(x, y)$  relating  $j(\tau)$  and  $j(N\tau)$ , and apply the same Galois invariance argument to it and its coefficients.

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